Propagation of e.m. waves in medium with finite μ, ε and σ

The Maxwell's equations are

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{——— (1)}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \text{——— (2)}$$

$$\nabla \cdot \vec{D} = \rho \quad \text{——— (3)}$$

and $\nabla \cdot \vec{B} = 0$ ——— (4)

Suppose $\sigma \neq 0$, i.e there may be a conduction current but the charge density is zero every where i.e $\rho = 0$ (charge can not reside inside the conductor although it may reside on its surface)

$$\nabla \cdot \vec{D} = 0 \implies \nabla \cdot (\varepsilon \vec{E}) = 0 \implies \nabla \cdot \vec{E} = 0$$

Taking curl of equation (1) we get

$$\nabla \times (\nabla \times \vec{E}) = -\nabla \times \frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \vec{B})$$

But $\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$

$$= 0 - \nabla^2 \vec{E}$$

$$\Rightarrow \quad -\nabla^2 \vec{E} = -\frac{\partial}{\partial t}(\nabla \times \vec{B})$$

$$\nabla^2 E = \mu \frac{\partial}{\partial t}(\nabla \times \vec{H}) \qquad [\because \vec{B} = \mu \vec{H}]$$

$$\nabla^2 E = \mu \frac{\partial}{\partial t}\left[\vec{J} + \frac{\partial \vec{D}}{\partial t}\right]$$

Also $\vec{J} = \sigma \vec{E}$ & $\vec{D} = \varepsilon \vec{E}$.

$$\nabla^2 \vec{E} = \mu \frac{\partial}{\partial t}\left[\sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}\right]$$

$$\nabla^2 E = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 E - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t} = 0 \quad \text{——— (5)}$$

This is general equation for electric vector in em wave in a medium.

Similarly for magnetic vector

$$\nabla^2 H - \mu\sigma \frac{\partial H}{\partial t} - \mu\varepsilon \frac{\partial^2 H}{\partial t^2} = 0 \quad \text{——— (6)}$$

- Equations (5) and (6) are the wave equations for electromagnetic fields $\vec{E}$ and $\vec{H}$ in a homogeneous isotropic conducting medium of conductivity σ, permittivity ε and permeability μ having no charge or current other than that determined by ohm's law.

Solutions of these equations for $\vec{E}$ and $\vec{H}$ are of the form

$$\vec{E} = \vec{E}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \quad \text{—(7)} \qquad k = \text{wave vector}$$

and $$\vec{H} = \vec{H}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \quad \text{—(8)}$$

~~from (5) and (7)~~

$$\cancel{\Rightarrow \nabla^2\left[\vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}\right] + \mu\sigma \frac{\partial \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}}{\partial t^2}}$$

$$\nabla^2\vec{E} = \nabla^2 \vec{E}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left[\vec{E}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}\right]$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left[\vec{E}_0\, e^{i(k_x x + k_y y + k_z z - \omega t)}\right]$$

$$= \vec{E}(-k_x^2) + \vec{E}(-k_y^2) + \vec{E}(-k_z^2)$$

$$\nabla^2\vec{E} = -\vec{E}(k_x^2 + k_y^2 + k_z^2) = -\vec{E}k^2 \quad \text{-----(9)}$$

$$\frac{\partial \vec{E}}{\partial t} = \frac{\partial}{\partial t}\left[\vec{E}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}\right] = \vec{E}(-i\omega) \quad \text{—(10)}$$

$$\frac{\partial^2 \vec{E}}{\partial t^2} = \frac{\partial^2}{\partial t^2}\left[\vec{E}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}\right] = \vec{E}(-i\omega)(-i\omega) = -\omega^2\vec{E} \quad \text{—(11)}$$

Substituting (9) (10) and (11) in (5)

$$-\vec{E}k^2 + i\mu\sigma\omega\vec{E} + \mu\varepsilon\omega^2\vec{E} = 0$$

$$\Rightarrow k^2 = i\mu\sigma\omega + \mu\varepsilon\omega^2$$

$$= i\frac{\mu\varepsilon\omega^2\sigma}{\varepsilon\omega} + \mu\varepsilon\omega^2$$

$$k^2 = \mu\varepsilon\omega^2\left[1 + \frac{i\sigma}{\varepsilon\omega}\right] \quad \text{—(11)}$$

Hence propagation vector is of complex form

$$\ldots\; A^2 - B^2 + 2iAB \quad \text{—(12)}$$

(11) and (12)

$$\Rightarrow \quad A^2 - B^2 = \mu\epsilon\omega^2$$

$$2AB = \mu\omega\sigma$$

$$B = \frac{\mu\omega\sigma}{2A}$$

$$\Rightarrow \quad A^2 - \frac{\mu^2\omega^2\sigma^2}{4A^2} = \mu\epsilon\omega^2$$

Put $A^2 = \alpha$

$$\alpha - \frac{\mu^2\omega^2\sigma^2}{4\alpha} = \mu\epsilon\omega^2$$

$$4\alpha^2 - 4\mu\epsilon\omega^2\alpha - \mu^2\omega^2\sigma^2 = 0$$

$$\alpha = \frac{4\mu\epsilon\omega^2 \pm \sqrt{16\mu^2\epsilon^2\omega^4 + 16\mu^2\omega^2\sigma^2}}{8}$$

$$A^2 = \frac{\mu\omega^2\epsilon + \mu\epsilon\omega^2\sqrt{1 + \frac{\sigma^2}{\epsilon^2\omega^2}}}{2}$$

$$A = \omega\sqrt{\mu\epsilon}\left[\frac{1 + \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2}}{2}\right]^{1/2}$$

similarly

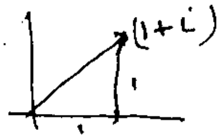
$$B = \omega\sqrt{\mu\epsilon}\left[\frac{\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1}{2}\right]^{1/2}.$$

$\frac{\sigma}{\epsilon\omega} >> 1$ for a good conductor

$$\therefore \quad A = \omega\frac{\sqrt{\mu\epsilon}}{\sqrt{2}}\left(\frac{\sigma}{\epsilon\omega}\right)^{1/2}$$

$$A = \sqrt{\frac{\omega\mu\sigma}{2}}$$

$$B = \sqrt{\frac{\omega\mu\sigma}{2}}$$



$$\Rightarrow \quad A = B.$$

$$\Rightarrow \quad \vec{k} = (A + iB)\hat{n} = A(1+i)\hat{n} = \sqrt{\frac{\omega\mu\sigma}{2}}\,(1+i)\,\hat{n}$$

$$\vec{E} = \vec{E}_0\, e^{i[(A+iB)\hat{n}\cdot\vec{r} - \omega t]}$$

$$\vec{E} = \vec{E}_0\, e^{i(A\hat{n}\cdot\vec{r} - \omega t)} \cdot e^{-B\hat{n}\cdot\vec{r}} \checkmark$$

Similarly

$$\vec{H} = \vec{H}_0\, e^{i(A\hat{n}\cdot\vec{r} - \omega t)} \cdot \underline{e^{-B\hat{n}\cdot\vec{r}}}$$ Damping part.