Maxwell's Equations

There are 4 maxwell's electromagnetic equations

1. Maxwells 1st equation or Gauss Law in electrostatics

$$\phi_E = \oint \vec{E}\cdot\vec{ds} = \frac{q}{\varepsilon_0}$$

This gives rise to $\nabla\cdot\vec{D} = \rho$.

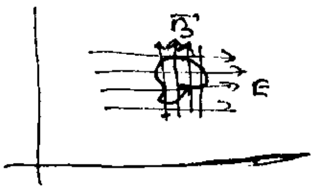
2. Maxwell's 2nd equation or Gauss Law in Magnetism:

$$\phi_B = \oint \vec{B}\cdot\vec{ds} = 0.$$

This gives rise to $\nabla\cdot\vec{B} = 0 \Rightarrow \nabla\cdot\vec{H} = 0$

3. Maxwell's 3rd Equation or Faraday's Law of electromagnetic Induction.

$$\oint \vec{E}\cdot\vec{dl} = -\frac{d}{dt}\int_S \vec{B}\cdot\vec{ds}$$



This gives rise to $\nabla\times\vec{E} = -\frac{\partial B}{\partial t}$

4. Maxwell's 4th Equation or Modified Ampere's Law

$$\oint \vec{H}\cdot\vec{dl} = I \qquad \text{where } I = \int_S \vec{J}\cdot\vec{ds}$$

or. $\oint \vec{B}\cdot\vec{dl} = \mu_0\oint \vec{H}\cdot\vec{dl} = \mu_0 I$ (in free space).

This gives rise to $\nabla\times\vec{H} = \vec{J} + \frac{\partial D}{\partial t}$.

* This Law [redacted] holds good for time varying fields also

Integeral Form of Maxwell's Equations :-

i) $\oint_S \vec{D}\cdot\vec{ds} = \int_V \rho\, dV$ ii) $\oint \vec{B}\cdot\vec{ds} = 0$

iii) $\oint \vec{E}\cdot\vec{dl} = -\int_S \frac{\partial\vec{B}}{\partial t}\cdot\vec{ds}$ iv) $\oint \vec{H}\cdot\vec{dl} = \int_S \left[\vec{J} + \frac{\partial D}{\partial t}\right]\cdot\vec{ds}$

Note:-(1) Gauss Divergence theorem : It is a mathematical theorem of convertion of surface integeral to volume integeral

$$\oint_S \vec{A}\cdot\vec{ds} = \oint_V \nabla\cdot\vec{A}\,dV \qquad \text{where } \vec{A} \text{ is any vector}$$

e.g

$$\oint_S \vec{E}\cdot\vec{ds} = \oint_V \nabla\cdot\vec{E}\,dV \qquad \nabla\cdot\vec{E} \Rightarrow \text{divergence of } \vec{E}$$

Note:-(2):— Stokes Theorem :- It is a theorem of convertion of line integeral to surface integeral.

$$\oint_l \vec{A}\cdot\vec{dl} = \int_S (\nabla\times\vec{A})\cdot\vec{ds}$$

e.g

$$\oint_l \vec{E}\cdot\vec{dl} = \int_S (\nabla\times\vec{E})\cdot\vec{ds}.$$

Differential form of Maxwell's Equations

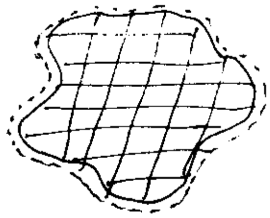
Maxwell's 1st Equation or Gauss Law in Electrostatics :

According to this law the electric flux passing through any closed surface (Gaussian surface) in an electric field is $\frac{1}{\varepsilon_0}$ times the total charge enclosed by the surface.
Also Electric flux linked with a given surface in electric field is given by surface integeral of electric field

$$\phi_E = \oint_S \vec{E}\cdot\vec{dS} = \frac{q}{\varepsilon_0} \quad \text{———} (1)$$

If ρ is volume charge density and q is the charge enclosed in a given volume V which itself is bound by surface S then

$$q = \int \rho \, dV \quad \text{———} (2)$$



By Divergence theorem

$$\oint_S \vec{E}\cdot\vec{dS} = \oint_V \nabla\cdot\vec{E}\, dV \quad \text{———} (3)$$

Equations (1) (2) and (3) imply that

$$\oint_V \nabla\cdot\vec{E}\, dV = \oint \frac{\rho\, dV}{\varepsilon_0}.$$

$$\left[\begin{array}{l} \oint \nabla\cdot\varepsilon_0 E\, dV = \oint \rho dV \\ \oint \nabla\cdot D\, dV = \oint \rho dV \end{array}\right]$$

$$\Rightarrow \quad \nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{———} (4)$$

For Time varying field Electric displacement vector

$\varepsilon_0 E = D$

$$(4) \Rightarrow \nabla\cdot\varepsilon_0\vec{E} = \rho$$

$$\Rightarrow \nabla\cdot\vec{D} = \rho.$$ This is differential form

Integeral form of Maxwell's equation

$$(3) \Rightarrow \oint_S \varepsilon_0 \vec{E}\cdot dS = \oint \nabla\cdot\varepsilon_0\vec{E}\, dV$$

$$\Rightarrow \oint_S \vec{D}\cdot\vec{dS} = \oint_V \nabla\cdot D\, dV = \int_V \rho\, dV$$ This is integeral form of Maxwell's 1st eqn.

Differential & Integeral form of Maxwell's 2nd equation or Gauss Law in Magnetism

It states that magnetic flux passing through closed surface placed in magnetic field is zero.

$$\phi_B = \oint_S \vec{B}\cdot\vec{ds} = 0 . \quad \text{——— (1)}$$

This equation tells that existence of free magnetic pole is not possible. This means that within a given volume bound by a closed surface (Gaussian Surface) only magnetic dipoles consisting of equal and opposite pole strength exist so that the net pole strength within the closed surface is zero.

From Gauss Divergence Theorem

$$\oint_S \vec{B}\cdot\vec{ds} = \oint_V \nabla\cdot\vec{B}\, dV \quad \text{——— (2)}$$

(1) and (2) $\Rightarrow \oint \nabla\cdot\vec{B}\, dV = 0$

$\Rightarrow \nabla\cdot\vec{B} = 0$. This is differential form

Equation (1) gives the integeral form of maxwells eqn.

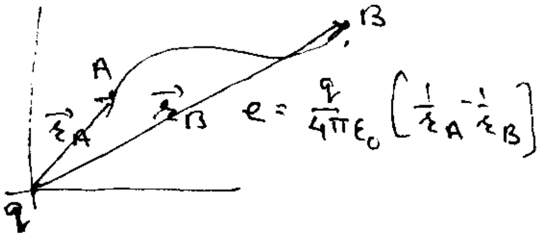
Maxwell's 3rd Equation or Faradays Law of electromagnetic induction (Differential & integeral form)

According to Faradays 2nd law emf induced in a closed conducting loop is directly proportional to the rate of change of magnetic flux.

$$e = -\frac{d\phi_B}{dt} \quad \text{——— 1)}$$

But

$$e = \oint \vec{E}\cdot\vec{dl} \quad \text{——— (2)}$$



$$\Rightarrow \oint \vec{E}\cdot\vec{dl} = -\frac{d\phi_B}{dt} \quad \text{- - - - - (2a)}$$

Also

$$\phi_B = \int_S \vec{B}\cdot\vec{ds} \quad \text{——— (3)}$$

$$\Rightarrow \oint \vec{E}\cdot\vec{dl} = -\frac{\partial}{\partial t}\int \vec{B}\cdot\vec{ds} .$$

Applying stokes theorem to equation (2)

$$\oint_l \vec{E}\cdot\vec{dl} = \int (\nabla\times\vec{E})\cdot\vec{ds} \quad \text{——— (4)}$$

(2a), (3) and (4) $\Rightarrow \int_S (\nabla\times\vec{E})\cdot\vec{ds} = -\frac{d}{dt}\int_S \vec{B}\cdot\vec{ds}$ ——— (5)

$$\Rightarrow \oint_S \nabla \times \vec{E} \cdot \vec{ds} = -\int \frac{\partial \vec{B}}{\partial t} \cdot \vec{ds} \qquad \text{——— (6)}$$

Equation (6) is valid for all surfaces

$$\Rightarrow \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{- - - - (7)}$$

Equation (7) is Differential form where as equation (4) is integeral form of maxwell's 3rd equation.

Integeral form of maxwell's 3rd equation states that line integeral of electric field around any closed path equals the time rate of change of magnetic field through the surface area bound by that path.

Differential form shows that curl of electric field over any closed path equals the time rate of change of magnetic field.

Maxwell's 4th Equation (Differential & integeral form):-

(Modified Ampere's Law)

According to Ampere circuital law the line integeral of magnetic field taken aloud any closed path is equal to current enclosed ~~by the~~ within closed path

$$\oint \vec{H}\cdot\vec{dl} = I \quad \text{——(1)}$$ H = Magnetic field strength

If the current is distributed over the entire space and $\vec{J}$ is the current density then

$$\oint \vec{H}\cdot\vec{dl} = \int_S \vec{J}\cdot\vec{dS} \quad \text{——(2)} \qquad \left(\because I = \int_S \vec{J}\cdot ds\right)$$

Applying stokes theorem

$$\oint \vec{H}\cdot\vec{dl} = \int_S (\nabla\times\vec{H})\cdot\vec{dS} \quad \text{——(3)}$$

from (2) and (3)

$$\int (\nabla\times\vec{H})\cdot\vec{dS} = \int_S \vec{J}\cdot\vec{dS}$$

$$\Rightarrow \quad \nabla\times\vec{H} = \vec{J} \quad \text{——————(4)}$$

However this law does not hold good for time varying fields.

Maxwell modified eqn (4) by introducing the concept of displacement current.

Taking Divergence of eqn (4)

$$\nabla\cdot(\nabla\times\vec{H}) = \nabla\cdot\vec{J} \quad \text{——(5)}$$

But divergence of curl of a vector is always zero

$$\therefore \quad 0 = \nabla\cdot\vec{J}$$

i.e $\nabla\cdot\vec{J}$ should be zero.

But from equation of continuity

$$\nabla\cdot\vec{J} = -\frac{\partial\rho}{\partial t} \quad \text{———} \quad (6) \Rightarrow -\frac{\partial\rho}{\partial t} = 0$$ This is true for static field but is not true for time varying field

Also from Gauss law of electrostatics (Maxwell 1st eqn)

$$\nabla\cdot\vec{D} = \rho \Rightarrow \nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{——(7)}$$

Differentiating partially w.r.t t

$$\nabla \cdot \frac{\partial E}{\partial t} = \frac{1}{\varepsilon_0} \frac{\partial \rho}{\partial t}$$

or $$\varepsilon_0 \nabla \cdot \frac{\partial E}{\partial t} = \frac{\partial \rho}{\partial t}$$

Adding $\nabla \cdot \vec{J}$ on both side

$$\nabla \cdot \vec{J} + \varepsilon_0 \nabla \cdot \frac{\partial E}{\partial t} = \nabla \cdot J + \frac{\partial \rho}{\partial t}$$

$$= -\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial t}$$

$$\Rightarrow \quad \nabla \cdot \vec{J} + \varepsilon_0 \nabla \cdot \frac{\partial E}{\partial t} = 0 .$$

$$\Rightarrow \quad \nabla \cdot \vec{J} + \nabla \cdot \frac{\partial D}{\partial t} = 0 \qquad (\varepsilon_0 E = D).$$

$$\nabla \cdot \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) = 0$$

$\vec{J} + \frac{\partial \vec{D}}{\partial t}$ is total current density

$\therefore$ Modified Ampere circuital law is

$$\boxed{\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}} \quad \text{——— (8)} \qquad \text{Differential form}$$

$\vec{J}$ = conduction current density

$\frac{\partial D}{\partial t}$ gives Displacement current so displacement current density J_d

from (3)

$$\oint \vec{H} \cdot \vec{dl} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot \vec{dS} \quad \text{——— (9)} \qquad \text{Integral form.}$$

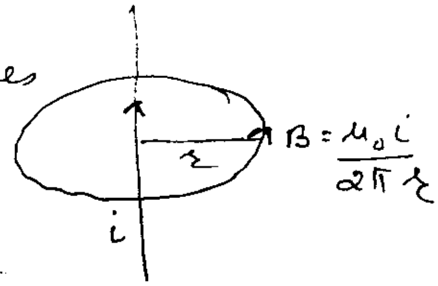
Equation (8) gives differential form where as equation (9) gives ~~....~~ integral form of 4th Maxwell's equation.

- Ampere's Law and displacement current

Amperes Law :- It states that line integral of magnetic field for any closed path in magnetic field is μ_0 times the current through the area enclosed by the closed path.

$$\Rightarrow \quad \oint \vec{B} \cdot \vec{dl} = \mu_0 i$$

Proof : If i = current in a straight conductor then magnetic field lines will be circular around the conductor.



At a distance r the magnetic field is given by

$$B = \frac{\mu_0 i}{2\pi r}$$

$\vec{B}$ and any small part $\vec{dl}$ of circular path are in same direction at every point.

$$\Rightarrow \oint \vec{B} \cdot \vec{dl} = \oint B\, dl = \oint \frac{\mu_0 i}{2\pi r} dl = \frac{\mu_0 i}{2\pi r} \oint dl$$

$$\Rightarrow \oint \vec{B} \cdot \vec{dl} = \frac{\mu_0 i}{2\pi r} \cdot l = \frac{\mu_0 i}{2\pi r} \times 2\pi r$$

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 i$$

→ Note:- The law is applicable to any number of currents passing through the area bound by the closed curve.