Poynting Theorem and Poynting vector

Poynting vector is the rate of flow of energy transmitted through unit area perpendicular to the direction of propagation of energy. It is represented by $\vec{P}$ or $\vec{S}$.

When e.m. waves travel in space from one pt to another pt. they transport electric and magnetic energy.

From Maxwell's equations

$$\nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} \quad \text{------(1)}$$

and

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \text{------(2)}$$

Taking dot product of equation (1) with $\vec{E}$ and of equation (2) with $\vec{H}$.

$$\vec{E}\cdot(\nabla\times\vec{H}) = \vec{E}\cdot\vec{J} + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} = \vec{E}\cdot\vec{J} + \epsilon\,\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}$$

$$\Rightarrow \vec{E}\cdot(\nabla\times\vec{H}) = \vec{E}\cdot\vec{J} + \frac{\partial}{\partial t}\left[\frac{1}{2}\epsilon E^2\right]^{*} \text{---(3)} \quad [\because D = \epsilon_0 E]$$

And

$$\vec{H}\cdot(\nabla\times\vec{E}) = -\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} = -\mu_0\vec{H}\cdot\frac{\partial\vec{H}}{\partial t}$$

$$= -\frac{\partial}{\partial t}\left[\frac{1}{2}\mu_0 H^2\right] \quad \text{----(4)}$$

(3) − (4) $\Rightarrow$

$$\vec{E}\cdot(\nabla\times\vec{H}) - \vec{H}\cdot(\nabla\times\vec{E})^{**} = \vec{E}\cdot\vec{J} + \frac{\partial}{\partial t}\left[\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right]$$

$$\Rightarrow -\nabla\cdot(\vec{E}\times\vec{H}) = \vec{E}\cdot\vec{J} + \frac{\partial}{\partial t}\left[\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2}\right]$$

integrating both sides over a volume V

$$\frac{\partial}{\partial t}\int_V \left(\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2}\right) dV + \int_V \vec{E}\cdot\vec{J}\, dV = -\int_V \nabla\cdot(\vec{E}\times\vec{H})\, dV \quad \text{---(5)}$$

From Divergence theorem.

$$\int_S (\vec{E}\times\vec{H})\cdot d\vec{S} = \int_V \nabla\cdot(\vec{E}\times\vec{H})\, dV$$

$$\Rightarrow \frac{\partial}{\partial t}\int_V \left(\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2}\right) dV + \int_V \vec{E}\cdot\vec{J}\, dV = -\int_S (\vec{E}\times\vec{H})\cdot d\vec{S}$$

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* $$\frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon_0 E^2\right) = \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon_0\, \vec{E}\cdot\vec{E}\right) = \frac{\epsilon}{2}\left[\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} + \frac{\partial\vec{E}}{\partial t}\cdot\vec{E}\right] = \frac{\epsilon_0}{2}\, 2\,\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}$$

$$= \epsilon_0 E$$

** $$\nabla\cdot(\vec{B}\times\vec{C}) = \vec{C}\cdot\nabla\times\vec{B} - \vec{B}\cdot\nabla\times\vec{C}$$

$$\Rightarrow \oint (\vec{E}\times\vec{H})\cdot d\vec{s} = -\frac{\partial}{\partial t}\int_V \left(\frac{\varepsilon E^2}{2} + \frac{\mu H^2}{2}\right) dV - \int \vec{E}\cdot\vec{J}\, dV \quad \text{—(6)}$$

~~The quantity $\vec{E}\times\vec{H}$ is called Pointing vector $\vec{P}$ or $\vec{S}$~~

$$\cancel{\Rightarrow \oint \vec{P}\cdot d\vec{s} = -\frac{\partial}{\partial t}\int_V \left(\frac{\varepsilon E^2}{2} + \frac{\mu H^2}{2}\right) dV + \int \vec{E}\cdot\vec{J}\, dV} \quad \text{—(7)}$$

- 1st term in equation (6) represents the rate of decrease of energy stored in a volume due to electric and magnetic field, whereas 2nd term represents the rate at which e.m. energy is lost through Joule heating.

Hence $\oint(\vec{E}\times\vec{H})\cdot ds$ represents the rate of flow of energy over the surface S enclosing volume V.

$\Rightarrow$ $\vec{E}\times\vec{H}$ gives the rate of flow of energy through unit area enclosing volume V and is denoted by $\vec{P}$.

$\therefore$ $\underline{\vec{P} = \vec{E}\times\vec{H}}$