HEISENBERG'S UNCERTAINTY PRINCIPLE

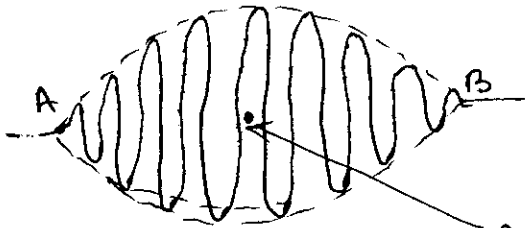
It states that it is impossible to determine the exact position and momentum of a particle simultaneously and that the product of uncertainty of momentum and uncertainty of position is $\geq h/2\pi$*

If Δp and Δx are uncertainties in the measurements of momentum and the position then

$$\Delta p \cdot \Delta x \geq h/2\pi \qquad (h/2\pi \text{ is denoted by } \hbar)$$

Proof of uncertainty principle: Heisenberg's uncertainty principle can be proved by assuming that a particle in motion can be taken as a group of waves and that the group velocity is equal to particle velocity

Moving particle must be considered as a de Broglie wave group rather than a localised entity.



Hence there must always be a limit to accuracy with which one can measure particle properties such as momentum & position or Energy and time etc.

Consider two waves of angular frequencies ω_1 and ω_2 and wave numbers K_1 and K_2. For these waves we have

$$\psi_1 = A \sin(\omega_1 t - K_1 x)$$

$$\psi_2 = A \sin(\omega_2 t - K_2 x)$$

From principle of superposition

Note: Also Energy and time can not be measured simultaneously and $\Delta E \cdot \Delta t \geq h/2\pi$.

Momentum and position & Energy and time are called canonically conjugate quantities.

$$\psi = \psi_1 + \psi_2$$

$$\psi = A\left[\sin(\omega_1 t - k_1 x) + \sin(\omega_2 t - k_2 x)\right] \quad \text{--- (1)}$$

Now $\sin C + \sin D = 2 \sin\frac{C+D}{2} \cos\frac{C-D}{2}$

$$\Rightarrow \psi = 2A \sin\left[\left(\frac{\omega_1+\omega_2}{2}\right)t - \frac{(k_1+k_2)x}{2}\right] \cos\left[\frac{(\omega_1-\omega_2)t}{2} - \frac{(k_1-k_2)x}{2}\right]$$

Since ω_1 and ω_2 are nearly equal

$\therefore \frac{\omega_1+\omega_2}{2} = \omega$ = mean angular frequency

$\frac{k_1+k_2}{2} = k$ = mean wave number

$\omega_2 - \omega_1 = \delta\omega$ = difference of angular freq

$k_2 - k_1 = \delta k$ = " " wave no.

$$\Rightarrow \psi = 2A \sin(\omega t - kx) \cos\left(\frac{\delta\omega}{2}.t - \frac{\delta k}{2}.x\right) \quad \text{--- (2)}$$

The resultant wave is shown in the figure in the begining.

The loop AB shown in figure moves with group velocity. Since group velocity is equal to particle velocity, therefore, the loop formed is equivalent to position of the particle.

The position of particle can not be given with certainty. It is some where between successive nodes A and B.

$\therefore \Delta x$ = error in measurement of the particle
= Distance between nodes.

we get nodes where ever

$$\cos\left(\frac{\delta\omega}{2}t - \frac{\delta k}{2}x\right) = 0 = \cos(2m+1)\pi/2 \qquad m = 0, 1, 2$$

$$\therefore \frac{\delta\omega}{2}t - \frac{\delta k.x}{2} = \pi/2,\ 3\pi/2,\ 5\pi/2 \ldots$$

If x_1 and x_2 are positions of two node at a given time t

then $\frac{\delta\omega}{2}t - \frac{\delta k}{2}x_1 = (2m+1)\pi/2$

& $\frac{\delta\omega}{2}t - \frac{\delta k}{2}x_2 = (2m+3)\pi/2$

$$\Rightarrow \frac{\delta k}{2}(x_1 - x_2) = \pi.$$

$$x_2 - x_1 = \frac{2\pi}{\Delta k} = \Delta x \quad \text{---} (3) \qquad \delta k \rightarrow \Delta k$$

$$\therefore \Delta x = \frac{2\pi}{\Delta\left(\frac{2\pi}{\lambda}\right)} = \frac{1}{\Delta\left(\frac{p}{h}\right)} = \frac{1}{\frac{1}{h}\Delta p} = \frac{h}{\Delta p}$$

$$\Delta p \, \Delta x = h \quad \text{---} (4)$$

$$\Rightarrow \quad \Delta p \, \Delta x \geq h/2\pi$$

$$\Delta p \, \Delta x \geq \hbar$$

Energy-Time uncertainty Principle:- It states that energy and time can not be measured simultaneously and the product of measurement in the uncertainties of energy & time is $\geq \hbar$

$$\Delta E \, \Delta t \geq \hbar$$

Proof

Let Δx be the uncertainty in measurement of x-coordinate.

If v_g is group velocity then uncertainty in measurement of time is given by

$$\Delta t = \frac{\Delta x}{v_g} \quad \text{---} (1)$$

Also $$\Delta E = \frac{\partial E}{\partial p_x} \cdot \Delta p_x \quad \text{---} (2)$$

Also $$E = \frac{p_x^2}{2m} \Rightarrow \frac{\partial E}{\partial p_x} = \frac{2p_x}{2m} = \frac{2mv_x}{2m} = v_x$$

$$\Rightarrow \frac{\partial E}{\partial p_x} = v_x = v_g$$

$$\Rightarrow \Delta E = v_g \Delta p_x \quad \text{---} (3)$$

$$(1) \,\&\, (3) \Rightarrow \Delta E \, \Delta t = \frac{\Delta x}{v_g} \times v_g \Delta p_x$$

$$= \Delta p_x \Delta x \geq \hbar$$

$$\therefore \underline{\Delta E \cdot \Delta t \geq \hbar}$$