Relation between ~~Group~~ Group velocity (v_g) and phase velocity (v_p):

Let two waves of frequencies ω and $\omega+\Delta\omega$, & wave numbers k and $k+\Delta k$ superimpose over each other so that

$$y_1 = A\cos(\omega t - kx)$$

and $$y_2 = A\cos[(\omega+\Delta\omega)t - (k+\Delta k)x]$$

$\therefore$ Net displacement at position x and time t is

$y = y_1 + y_2$ (Principle of superposition)

$$\Rightarrow y = A\left[\cos[(\omega+\Delta\omega)t - (k+\Delta k)x] + \cos(\omega t - kx)\right]$$

Since $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

$$\therefore y = 2A\cos\left[\left(\omega+\frac{\Delta\omega}{2}\right)t - \left(k+\frac{\Delta k}{2}\right)x\right]\cos\left(\frac{\Delta\omega t}{2} - \frac{\Delta k . x}{2}\right)$$

$$\Rightarrow y = 2A\cos(\omega t - kx)\cos\left(\frac{\Delta\omega t}{2} - \frac{\Delta k . x}{2}\right)$$

$$\begin{bmatrix} \Delta\omega << \omega \\ \Delta k << k \end{bmatrix}$$

$$y = 2A\cos\left(\frac{\Delta\omega . t - \Delta k x}{2}\right)\cos(\omega t - kx) \quad \text{——— (1)}$$

This is equation of resulting wave packet

~~The term $2A\cos\left(\frac{\Delta\omega . t - \Delta k . x}{2}\right)$ gives the amplitude of group of waves (wave packet)~~

Amplitude of wave packet is $2A\cos\left(\frac{\Delta\omega . t - \Delta k x}{2}\right)$ and its phase is $\omega t - kx$. Since superposition of waves does not change the phase

$$\therefore \omega t - kx = \text{constant} \quad \text{——— (2)}$$

$\therefore \frac{dx}{dt}$ gives phase velocity and is obtained by differentiating (2) w.r.t. t

$$\Rightarrow \omega - k\frac{dx}{dt} = 0 \Rightarrow \omega = k\frac{dx}{dt} = k . v_p$$

$$\Rightarrow v_p = \frac{\omega}{k} \quad \text{——— (3)}$$

For amplitude of wave packet to be constant

$$\frac{\Delta\omega . t - \Delta k . x}{2} = \text{constant}$$

$$\Rightarrow \quad \Delta\omega . t - \Delta k . x = \text{constant} \quad \text{——— (4)}$$

In this case $v_g = \frac{dx}{dt}$ is obtained by differentiating (4)

$$\Rightarrow \quad \Delta\omega - \Delta k \frac{dx}{dt} = 0$$

$$\frac{dx}{dt} = \frac{\Delta\omega}{\Delta k} \Rightarrow v_g = \frac{\Delta\omega}{\Delta k} \quad \text{——— (5)}$$

If changes in ω and k are very small ie $\to 0$ then

$$v_g = \frac{d\omega}{dk} \quad \text{——— (6)}$$

$$v_g = \frac{d}{dk}(k v_p) \qquad (\because \omega = k v_p)$$

$$= k . \frac{dv_p}{dk} + v_p$$

$$= v_p + \frac{2\pi}{\lambda} \frac{dv_p}{d\left(\frac{2\pi}{\lambda}\right)}$$

$$= v_p + \frac{2\pi}{\lambda} \frac{1}{\left(\frac{-2\pi}{\lambda^2}\right)} . \frac{dv_p}{d\lambda}$$

$$v_g = v_p - \lambda \frac{dv_p}{d\lambda} \quad \text{——— (7)}$$

$$\Rightarrow \quad v_g < v_p$$

<u>Different cases</u>

i) If $\frac{dv_p}{d\lambda} = 0$ the <u>medium is called nondispersive medium</u>

$$\Rightarrow \quad v_g = v_p$$

ie Group velocity is equal to phase velocity

ii) If $\frac{dv_p}{d\lambda} > 0$ the medium is called dispersive medium and group velocity will be less than phase velocity

Phase velocity $v_p = \frac{\omega}{k}$

But $\omega = 2\pi\nu = \frac{2\pi mc^2}{h} = \frac{2\pi c^2}{h} \cdot \frac{m_0}{\sqrt{1-v^2/c^2}}$ —— (8)

and $k = \frac{2\pi}{\lambda} = \frac{2\pi p}{h} = \frac{2\pi mv}{h} = \frac{2\pi v}{h} \cdot \frac{m_0}{\sqrt{1-v^2/c^2}}$ —— (9)

$\Rightarrow v_p = \frac{c^2}{v}$ —— (10)

Also group velocity

$v_g = \frac{d\omega}{dk} = \frac{d\omega/dv}{dk/dv}$ —— (11)

(8) $\Rightarrow \frac{d\omega}{dv} = \frac{2\pi c^2 m_0}{h} \frac{d}{dv}\left(1-\frac{v^2}{c^2}\right)^{-1/2}$

$= -\frac{1}{2} \cdot \frac{2\pi c^2 m_0}{h} \left(1-\frac{v^2}{c^2}\right)^{-3/2} \left(-\frac{2v}{c^2}\right)$

$\frac{d\omega}{dv} = \frac{2\pi v m_0}{h} \left(1-\frac{v^2}{c^2}\right)^{-3/2}$ —— (12)

(9) $\Rightarrow \frac{dk}{dv} = \frac{2\pi m_0}{h} \frac{d}{dv}\left[v\left(1-\frac{v^2}{c^2}\right)^{-1/2}\right]$ —— (13)

$= \frac{2\pi m_0}{h}\left[v \cdot \left(-\frac{1}{2}\right)\left(1-\frac{v^2}{c^2}\right)^{-3/2}\left(-\frac{2v}{c^2}\right) + \left(1-\frac{v^2}{c^2}\right)^{-1/2}\right]$

$= \frac{2\pi m_0}{h}\left(1-\frac{v^2}{c^2}\right)^{-3/2}\left[\frac{v^2}{c^2} + \left(1-\frac{v^2}{c^2}\right)\right]$

$\frac{dk}{dv} = \frac{2\pi m_0}{h}\left(1-\frac{v^2}{c^2}\right)^{-3/2}$ —— (14)

(11) (12) and (14)

$\Rightarrow v_g = v$ —— (15)

Also $v_p v_g = \frac{c^2}{v} \times v$

$v_p v_g = c^2$

Hence deBroglie wave group or wave packet associated with the moving particle travels with same velocity as the moving particle.

Phase velocity is larger than particle velocity as well as velocity of light. However v_p has no meaning