Wave function :- Any wave is formed by some variation.

In case of sound wave, the pressure of gas changes periodically. In case of water waves, the height of water surface changes periodically.

In case of electromagnetic wave, the electric and magnetic fields vary periodically.

In case of matter wave, it is wave function which changes. The changes in wave function give rise to matter wave.

A wave function ψ contains a real part A and imaginary part B

$$\Rightarrow \quad \psi = A + iB$$

Conjugate of ψ is denoted by ψ^* and is given by

$$\psi^* = A - iB.$$

$$\psi\psi^* = A^2 + B^2$$

Also $|\psi|^2 = A^2 + B^2$

$$\Rightarrow \psi\psi^* = |\psi|^2 \quad \text{——— (1)}$$

Let us assume that the wave function is

$$\psi = A e^{-i\omega\left(t - \frac{x}{v}\right)} \quad \text{——— (2)}$$

where $\omega = 2\pi\nu$ & $v = \nu\lambda$.

$$\Rightarrow \quad \psi = A e^{-i2\pi\nu\left(t - \frac{x}{\nu\lambda}\right)}$$

$$\psi = A e^{-2\pi i\left(\nu t - \frac{x}{\lambda}\right)} \quad \text{——— (3)}$$

Now $E = h\nu = 2\pi\hbar\nu$ and $\lambda = \frac{h}{p} = \frac{2\pi\hbar}{p}$

$$\Rightarrow \psi = A e^{-2\pi i\left(\frac{E}{2\pi\hbar}t - \frac{xp}{2\pi\hbar}\right)}$$

$$\boxed{\psi = A e^{-\frac{i}{\hbar}[Et - px]}}$$

This is wave function for free particle

Energy operator

we know that

$$\psi = A e^{-\frac{i}{\hbar}[Et - px]}$$

Taking partial differential w.r.t t

$$\frac{\partial \psi}{\partial t} = \frac{-iEA}{\hbar} e^{-\frac{i}{\hbar}[Et-px]}$$

$$\frac{\partial \psi}{\partial t} = -\frac{iE}{\hbar}\psi \Rightarrow E\psi = -\frac{\hbar}{i}\frac{\partial \psi}{\partial t}$$

$$\Rightarrow E\psi = i\hbar \frac{\partial \psi}{\partial t}$$

$$\Rightarrow E = i\hbar \frac{\partial}{\partial t}$$

The operator $i\hbar \frac{\partial}{\partial t}$ is called energy operator

Momentum operator :-

we know that

$$\psi = A e^{-i/\hbar (Et - px)}$$

Differentiating partially w.r.t x

$$\frac{\partial \psi}{\partial x} = \frac{ip}{\hbar} A e^{-i/\hbar (Et - px)}$$

$$\frac{\partial \psi}{\partial x} = \frac{ip}{\hbar}\psi$$

$$\Rightarrow p\psi = \frac{\hbar}{i}\frac{\partial \psi}{\partial x}$$

$$\Rightarrow p = \frac{\hbar}{i}\frac{\partial}{\partial x}$$

The operator $\frac{\hbar}{i}\frac{\partial}{\partial x}$ is called momentum operator.

Normalisation of wave function:

i) If a ~~[illegible]~~ particle exists then the normalisation means

$$\int_{-\infty}^{+\infty} \psi\psi^* \, dV = 1 \Rightarrow \int_{-\infty}^{+\infty} |\psi|^2 dV = 1$$

This is because probability of finding the particle somewhere in whole space must be 1.

Note: whole space means $x = \pm\infty$ $y = \pm\infty$ $z = \pm\infty$

ii) If a particle does not exist then $\int_{-\infty}^{+\infty}|\psi|^2 dV = 0$

Note :- i. If $\int_{-\infty}^{+\infty}|\psi|^2 dV = 0$ then particle does not exist and ψ also does not exist.

ii) If $\int_{-\infty}^{+\infty}|\psi|^2 dV = 1$ then particle exists and wave function is acceptable.

iii) If a wave function ψ exists then it is acceptable if and only if $\int_{-\infty}^{+\infty}|\psi|^2 dV = 1$.

Requirements for a physically acceptable wave function:

1. ψ must be continuous and single valued function every where
2. $\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial z}$ must be continuous and single valued every where
3. wave function must be normalised i.e

$$\int_{-\infty}^{+\infty}|\psi|^2 dV = 1.$$

Probability of a particle to exist between limits x_1, x_2, y_1, y_2 & z_1, z_2

$$P_{x_1,x_2} = \int_{x_1}^{x_2}|\psi|^2 dx$$

$$P_{y_1,y_2} = \int_{y_1}^{y_2}|\psi|^2 dy$$

& $$P_{z_1,z_2} = \int_{z_1}^{z_2}|\psi|^2 dz.$$

Expectation Value :. Expectation value along x-direction is defined as

$$\langle x \rangle = \frac{\int_{-\infty}^{+\infty} x|\Psi|^2 dx}{\int_{-\infty}^{+\infty} |\Psi|^2 dx} = \frac{\int_{-\infty}^{+\infty} \Psi^* x \Psi \, dx}{\int_{-\infty}^{+\infty} \Psi^* \Psi \, dx}.$$

Since for a normalised wave function $\int_{-\infty}^{+\infty} \Psi^* \Psi \, dx = 1$.

$$\therefore \langle x \rangle = \int_{-\infty}^{+\infty} \Psi^* x \Psi \, dx = \int_{-\infty}^{\infty} x|\Psi|^2 dx$$